Regular Surfaces

Regular surfaces, Inverse images of regular values

Def. A subset $S \subseteq \mathbb{R}^3$ is a regular surface, if $\forall\, p \in S$, $\exists$ a nbd. $V \subseteq \mathbb{R}^3$ and a map

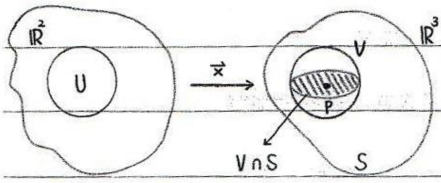
$\vec{x} : U \subseteq \mathbb{R}^2 \to V \cap S$ of an open set $U \subseteq \mathbb{R}^2$ onto $V \cap S \subseteq \mathbb{R}^3$ st.

(1) $\vec{x}$ is differentiable, i.e. $\vec{x}(u,v) = (x(u,v), y(u,v), z(u,v))$ all partial derivative of $\vec{x}$ are continuous

(2) $\vec{x}$ is homeomorphism, i.e. $\vec{x}$ is 1-1, onto, continuous and $\vec{x}^{-1}$ exists, continuous

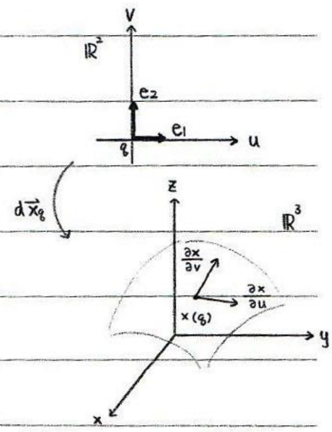
(3) $\vec{x}$ satisfies the regularity condition, i.e. $\forall\, q \in U$, the differential $d\vec{x}$, $d\vec{x} : \mathbb{R}^2 \to \mathbb{R}^3$ is 1-1.

(we want the map $d\vec{x}$ takes the unit vector in U into the tangent vector in $V \cap S$)



Here the mapping $\vec{x}$ is called parametrization

$V \cap S$ is called coordinate (local) nbd.



The vector e_1 is tangent vector to the curve $r_1 : u \to (u, v_0)$

The image curve $\alpha(u) = \vec{x}(u, v_0)$, $\alpha'(u) = d\vec{x}_q(e_1) = \frac{\partial \vec{x}}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$

The vector e_2 is tangent vector to the curve $r_2 : v \to (u_0, v)$

The image curve $\beta(v) = \vec{x}(u_0, v)$, $\beta'(v) = d\vec{x}_q(e_2) = \frac{\partial \vec{x}}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$

consider $r(t) = (at + u_0, bt + v_0)$, $r(0) = (u_0, v_0) = q$, $r'(t) = (a, b) = w$

The image curve $\beta(t) = \vec{x}(at + u_0, bt + v_0)$, $\beta'(t) = \frac{\partial \vec{x}}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial \vec{x}}{\partial v}\frac{\partial v}{\partial t} = a\frac{\partial \vec{x}}{\partial u} + b\frac{\partial \vec{x}}{\partial v}$

$d\vec{x}_q(ae_1 + be_2) = a\frac{\partial \vec{x}}{\partial u} + b\frac{\partial \vec{x}}{\partial v} = a\, d\vec{x}_q(e_1) + b\, d\vec{x}_q(e_2)$ $\quad \therefore d\vec{x}_q$ is a linear map, $d\vec{x}_q = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} = A$

$d\vec{x}_q$ is 1-1 $\Longleftrightarrow \frac{d\vec{x}}{du} \times \frac{d\vec{x}}{dv} \neq 0$, i.e. at least one of minors of A is nonzero

$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)}, \frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(x,z)}{\partial(u,v)}$ is nonzero, where $\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$

Def. Given a differentiable map $F: U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ defined in an open set U of $\mathbb{R}^n$.

We say $p \in U$ is a critical point of F is a differential $dF_p: \mathbb{R}^n \to \mathbb{R}^m$ is not a surjective map.

The image $F(p) \in \mathbb{R}^m$ of a critical point is called a critical value of F.

A point of $\mathbb{R}^m$ which is not a critical value is called a regular value of F.

$f: U \subseteq \mathbb{R}^3 \to \mathbb{R}$, $\{e_1, e_2, e_3\}$ is a basis for $\mathbb{R}^3$, $df(e_1) = f_x$, $df(e_2) = f_y$, $df(e_3) = f_z$

$a \in f(U)$ is a regular value $\Leftrightarrow$ f_x, f_y, f_z do not vanish simultaneously at any point in the inverse image $f^{-1}(a) = \{(x,y,z) \in U \mid f(x,y,z) = a\}$

Ex: $S^2 = \{(x,y,z) \mid x^2+y^2+z^2 = 1\}$, S^2 center $(0,0,0)$, radius 1

let $f(x,y,z) = x^2+y^2+z^2$, f is C^∞, $f_x = 2x$, $f_y = 2y$, $f_z = 2z$ $\Rightarrow$ $f_x = 0, f_y = 0, f_z = 0 \Leftrightarrow x = y = z = 0$

$(0,0,0)$ is a critical point of f $\Rightarrow$ 0 is a critical value of f, $(0,0,0) \notin f^{-1}(1) = \{(x,y,z) \mid x^2+y^2+z^2 = 1\}$

1 is a regular value of f, by Prop 2., $f^{-1}(1) = S^2$ is a regular surface

Prop 2. If $f: U \subseteq \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function, a is a regular value of f, then $f^{-1}(a)$ is a regular surface.

(Hope $f^{-1}(a) \equiv$ the graph of a differentiable function)

<pf> let $p = (x_0, y_0, z_0) \in f^{-1}(a)$, where a is a regular value of f. we may assume $\frac{\partial f}{\partial z}(p) \neq 0$

Define $F: U \subseteq \mathbb{R}^3 \to \mathbb{R}^3$, $F(x,y,z) = (x, y, f(x,y,z))$, $\det(dF_p) = \det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ f_x & f_y & f_z \end{pmatrix} = f_z|_p = \frac{\partial f}{\partial z}(p) \neq 0$

By IFT, $\exists$ nbd V of p and nbd W of $F(p)$ st. $F: V \to W \subseteq \mathbb{R}^3$ is differentiable, F^{-1} exists and diff.

let $F^{-1}(u,v,t) = (u, v, g(u,v,t))$ $\therefore x = u$, $y = v$, $z = g(u,v,t)$

In particular, $z = g(u,v,t)|_{t=a}$ is C^∞ and define $h(u,v) = g(u,v,t)|_{t=a} = g(u,v,a)$, h is C^∞

$F^{-1}(W \cap \{(u,v,t) | t=a\}) = V \cap \{(u,v,g(u,v,t)) | t=a\} = V \cap \{(u,v,h(u,v))\} \equiv$ graph of h $\equiv f^{-1}(a) \cap V$

$\because p \in f^{-1}(a)$ $\therefore$ By Prop 1, $f^{-1}(a) \cap V$ is a regular surface ($\because f^{-1}(a) \cap V \equiv$ graph of h)

Since p is arbitary in $f^{-1}(a)$ $\therefore f^{-1}(a)$ is a regular surface.

Def. A regular surface S is said orientable, if it is possible to cover with a family of coordinate nbd in such a way that if $p \in S$ belongs to 2 nbd of these families, then the change coordinate has positive Jacobian at p.

This choice of such a family is called orientation for S, and S in the case is called oriented.

If such a choice is not possible, then S is called nonorientable.

Ex: (1) S can be covered by only one parametrization, then S is orientable (graph of f)

(2) S is a regular surface and S can be covered by two parametrizations (X, U), $(\bar{X}, \bar{U})$ and $X(U) \cap \bar{X}(\bar{U})$ is connected, then S is orientable.

<sol> on $X(U)$, orientation $\{X_u, X_v\}$; on $\bar{X}(\bar{U})$, orientation $\{\bar{X}_{\bar{u}}, \bar{X}_{\bar{v}}\}$

If $\frac{\partial(u,v)}{\partial(\bar{u},\bar{v})} > 0$ (v) If $\frac{\partial(u,v)}{\partial(\bar{u},\bar{v})} < 0$, then switch $\frac{\partial(u,v)}{\partial(\bar{v},\bar{u})} > 0$, $\{\bar{X}_{\bar{u}}, \bar{X}_{\bar{v}}\} \to \{\bar{X}_{\bar{v}}, \bar{X}_{\bar{u}}\}$

(3) unit normal vector field on S, $X: U \to S$ parametrization, $\bar{X}: \bar{U} \to S$ another parametrization

$N: S \to \mathbb{R}^3$, $N(P) = \frac{X_u \times X_v}{|X_u \times X_v|}$ normal to $T_p(S)$, $p \in X(U)$; $\bar{N}(P) = \frac{\bar{X}_{\bar{u}} \times \bar{X}_{\bar{v}}}{|\bar{X}_{\bar{u}} \times \bar{X}_{\bar{v}}|}$

$N(P) = \bar{N}(P) \iff \frac{\partial(u,v)}{\partial(\bar{u},\bar{v})} > 0$

Prop. S is orientable $\iff$ S has a unit normal vector field which is differentiable (conti)

<pf> ($\Rightarrow$) If S is orientable, we can pick a family of parametrization, namely $\{x_i, U_i\}_{i \in I}$ st.

$S = \bigcup_{i \in I} x_i(U_i)$ and $\frac{\partial(u_i, v_i)}{\partial(u_j, v_j)} > 0$, $i \neq j$

on $x_i(U_i)$, define $N(P) = \frac{(x_i)_u \times (x_i)_v}{|(x_i)_u \times (x_i)_v|}$ normal to $T_p(S)$, differentiable. $\because$ x_i is differentiable.

$\therefore$ $N(P)$ is diff. unit normal vector field of S.

($\Leftarrow$) choose a family of connected coordinate nbd $\{x_i(U_i)\}_{i \in I}$ to cover S

$N(P) = \pm \frac{(x_i)_u \times (x_i)_v}{|(x_i)_u \times (x_i)_v|}$ on $x_i(U_i) = \begin{cases} + , \text{ keep} \\ - , \text{ change } u \leftrightarrow v \end{cases}$ If $x_i(U_i) \cap x_j(U_j) \neq \emptyset$, for $i \neq j$

claim: $\frac{\partial(u_i, v_i)}{\partial(u_j, v_j)} > 0$, set $p \in x_i(U_i) \cap x_j(U_j)$, $N(P) = \frac{(x_i)_u \times (x_i)_v}{|(x_i)_u \times (x_i)_v|} = \frac{(x_j)_u \times (x_j)_v}{|(x_j)_u \times (x_j)_v|}$

Fact : A regular surface may not be connected.

Def. A surface $S \subseteq \mathbb{R}^3$ is said to be connected, if any two of its points can be joined by a conti. curve in S

Prop 3. A regular surface S is locally graph of a differentiable function.

($\forall p \in S$, $\exists$ nbd V of p in S st. V ≡ the graph of differentiable function)

<pf> S is a regular surface, $\exists \vec{x} : U \subseteq \mathbb{R}^2 \to S \subseteq \mathbb{R}^3$ which is a parametrization of S, $\vec{x}(u,v) = (x, y, z)$

Let $q \in \vec{x}^{-1}_{(p)}$, in particular, we may assume $\frac{\partial(x,y)}{\partial(u,v)}(q) \neq 0$, consider projection map $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ as $\pi(x,y,z) = (x,y)$

$\pi \circ \vec{x} : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, $\pi \circ \vec{x}(u,v) = (x(u,v), y(u,v))$, $\det(d(\pi \circ \vec{x}))|_q = \frac{\partial(x,y)}{\partial(u,v)}|_q \neq 0$ $\therefore \pi \circ \vec{x}$ is isomorphism

By IFT, $\exists$ nbd V_1 of q, V_2 of $\pi \circ \vec{x}(q)$ st. $\pi \circ \vec{x} : V_1 \to V_2$ is C^∞ and $(\pi \circ \vec{x})^{-1}$ exists, C^∞

$\therefore (\pi \circ \vec{x})^{-1} : V_2 \to V_1$, $(\pi \circ \vec{x})^{-1}(x,y) = (u(x,y), v(x,y))$ $\quad * \ (\pi \circ \vec{x})^{-1} = \vec{x}^{-1} \circ \pi^{-1}$

$\therefore z(u(x,y), v(x,y))$ is a function on V_2 st. a piece of the surface can be represented the graph of z.

$(x(u,v), y(u,v), z(u,v))$ can rewrite as $(x, y, z(u(x,y), v(x,y)))$

Change of parameter, Differentiable function on surface

Def. S is a regular surface, $F : S \to \mathbb{R}$ is differentiable, if $\forall \vec{x}$ of S st. $F \circ \vec{x} : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ is differentiable.

Note. If point p belongs to two coordinate nbd with parameter (u,v) and $(\tilde{u}, \tilde{v})$,

it is possible to pass from one of two pairs of coordinate to the other by a diff. transformation

Prop. (Change of parameters) Let $p \in S$, S is regular surface in $\mathbb{R}^3$.

Let $\vec{x} : U \subseteq \mathbb{R}^2 \to S \subseteq \mathbb{R}^3$, $\vec{Y} : V \subseteq \mathbb{R}^2 \to S \subseteq \mathbb{R}^3$ be two parametrization of S st. $p \in \vec{x}(U) \cap \vec{Y}(V) = W$

then the change of coordinates $h = \vec{x}^{-1} \circ \vec{Y} : \vec{Y}^{-1}(W) \to \vec{x}^{-1}(W)$ is a diffeomorphism.

Application of 1st fundamental form

Determine the curve which makes a constant angle β with meridiams $= \Phi =$ constant

<sol> $\alpha(t) = (\theta(t), \phi(t))$, $\alpha'(t) = (\theta'(t), \phi'(t)) = \theta' \vec{X_\theta} + \phi' \vec{X_\phi}$, meridiam (θ, ϕ_0) fixed

Tangent vector of meridiam at point (θ, ϕ_0) will be $\theta' \vec{X_\theta}(\theta, \phi) = \vec{X_\theta}(\theta, \phi)$ $\because \theta' = 1$

$\langle \alpha'(t), \vec{X_\theta} \rangle = |\alpha'(t)||\vec{X_\theta}| \cos\beta \Rightarrow \cos\beta = \dfrac{\langle \alpha'(t), \vec{X_\theta} \rangle}{|\alpha'(t)||\vec{X_\theta}|}$ $\because \langle \alpha'(t), \vec{X_\theta} \rangle = \theta'$

$\langle \alpha'(t), \alpha'(t) \rangle = \langle \theta' \vec{X_\theta} + \phi' \vec{X_\phi}, \theta' \vec{X_\theta} + \phi' \vec{X_\phi} \rangle = (\theta')^2 E + 2\theta'\phi' F + (\phi')^2 G = (\theta')^2 + (\phi')^2 \sin^2\theta$

$\cos\beta = \dfrac{\theta'}{\sqrt{(\theta')^2 + (\phi')^2 \sin^2\theta}} \Rightarrow (\theta')^2 = \cos^2\beta\,[(\theta')^2 + (\phi')^2 \sin^2\theta] \Rightarrow (1 - \cos^2\beta)(\theta')^2 = \cos^2\beta \sin^2\theta\,(\phi')^2$

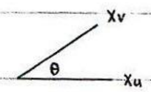
$\Rightarrow \sin^2\beta\,(\theta')^2 = \cos^2\beta \sin^2\theta\,(\phi')^2 \Rightarrow \tan^2\beta\,(\theta')^2 = \sin^2\theta\,(\phi')^2 \Rightarrow \csc\theta \cdot \theta' = \cot\beta \cdot \phi'$

let $\theta' dt = d\theta$, $\phi' dt = d\phi \Rightarrow \int \csc\theta\, d\theta = \pm \int \cot\beta\, d\phi \Rightarrow \ln|\csc\theta - \cot\theta| = \pm \cot\beta\, \phi + C = \ln \tan\frac{\theta}{2}$

Area of Surface

Def. $\vec{x} : U \subseteq \mathbb{R}^2 \to S$ is a parametrization of S, S is a regular surface. Q is an open set in $U \subseteq \mathbb{R}^2$.

Define $A(R) = \iint_Q |\vec{x_u} \times \vec{x_v}|\, du\, dv$, where $R = \vec{x}(Q)$



$|\vec{X_u} \times \vec{X_v}|^2 = |\vec{X_u}|^2 |\vec{X_v}|^2 \sin^2\theta$, $\langle \vec{X_u}, \vec{X_v} \rangle^2 = |\vec{X_u}|^2 |\vec{X_v}|^2 \cos^2\theta$

$|\vec{X_u} \times \vec{X_v}|^2 + \langle \vec{X_u}, \vec{X_v} \rangle^2 = |\vec{X_u}|^2 |\vec{X_v}|^2 \Rightarrow |\vec{X_u} \times \vec{X_v}|^2 = |\vec{X_u}|^2 |\vec{X_v}|^2 - \langle \vec{X_u}, \vec{X_v} \rangle^2 \Rightarrow |\vec{X_u} \times \vec{X_v}| = \sqrt{EG - F^2}$

Note. It is independent of $\vec{x}$ for $A(R)$, i.e. if $\vec{\bar{x}} : \bar{U} \to S$ is another parametrization of S st. $\vec{\bar{x}}(\bar{Q}) = R = \vec{x}(Q)$

<pf> $\iint_{\bar{Q}} |\vec{X_{\bar{u}}} \times \vec{X_{\bar{v}}}|\, d\bar{u}\, d\bar{v} \overset{?}{=} \iint_Q |X_u \times X_v|\, du\, dv$, consider $\Phi : x^{-1} \circ \bar{x} : \bar{U} \to U$

$$\begin{array}{l} X : U \searrow \\ \uparrow \Phi \qquad\quad S \\ \bar{X} : \bar{U} \nearrow \end{array}$$

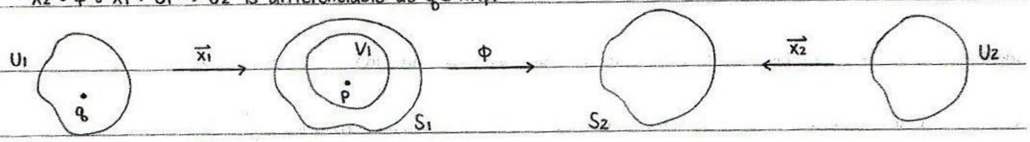
$\dfrac{\partial(u,v)}{\partial(\bar{u},\bar{v})} = \begin{pmatrix} \frac{\partial u}{\partial \bar{u}} & \frac{\partial u}{\partial \bar{v}} \\ \frac{\partial v}{\partial \bar{u}} & \frac{\partial v}{\partial \bar{v}} \end{pmatrix}$ is the Jacobian of change parameters Φ from $\bar{U}$ to U.

$\overline{X_{\bar{u}}} = \dfrac{\partial x}{\partial u}\dfrac{\partial u}{\partial \bar{u}} + \dfrac{\partial x}{\partial v}\dfrac{\partial v}{\partial \bar{u}} = X_u \dfrac{\partial u}{\partial \bar{u}} + X_v \dfrac{\partial v}{\partial \bar{u}}$, $\overline{X_{\bar{v}}} = \dfrac{\partial x}{\partial u}\dfrac{\partial u}{\partial \bar{v}} + \dfrac{\partial x}{\partial v}\dfrac{\partial v}{\partial \bar{v}} = X_u \dfrac{\partial u}{\partial \bar{v}} + X_v \dfrac{\partial v}{\partial \bar{v}}$

$\iint_{\bar{Q}} |\overline{X_{\bar{u}}} \times \overline{X_{\bar{v}}}|\, d\bar{u}\, d\bar{v} = \iint_{\bar{Q}} |X_u \times X_v| \left|\dfrac{\partial(u,v)}{\partial(\bar{u},\bar{v})}\right| d\bar{u}\, d\bar{v} = \iint_Q |X_u \times X_v|\, du\, dv$

Cor. A continuous map $\phi : V_1 \subseteq S_1 \to S_2$ of an open set V_1 of a regular surface S_1 to a regular surface S_2 is diff at $p \in V_1$, if given two parametrization $\vec{x_1} : U_1 \subseteq \mathbb{R}^2 \to S_1$, $\vec{x_2} : U_2 \subseteq \mathbb{R}^2 \to S_2$ with $p \in \vec{x_1}(U_1)$, $\phi(\vec{x_1}(U_1)) \subseteq \vec{x_2}(U_2)$

$\vec{x_2}^{-1} \circ \phi \circ \vec{x_1} : U_1 \to U_2$ is differentiable at $q \in \vec{x_1}^{-1}(p)$



Def. Two regular surfaces are diffeomorphism, if a diff. map $\Phi : S_1 \to S_2$ has a differential inverse.

Note. Any regular surface is locally diffeomorphism to an open set in $\mathbb{R}^2$

i.e. given $p \in S$, $\exists$ nbd V of p st. V is diffeomorphic to an open $U \subseteq \mathbb{R}^2$. ($\exists\, \Phi : V \to U$ diffeomorphism)

Ex: Let S be symmetric with respect to X-Y plane, i.e. $(x, y, z) \in S \Rightarrow (x, y, -z) \in S$

Define $\Phi : S \to S$, $\Phi(x, y, z) = (x, y, -z)$. Φ is diffeomorphism

Ex: Let $R_{z,\theta} : \mathbb{R}^3 \to \mathbb{R}^3$ be a rotation of angle θ about z-axis.

Let $S \subseteq \mathbb{R}^3$ st. S is invarient under the rotation $R_{z,\theta}$.

Let $p \in S$. $\Phi = R_{z,\theta} : S \to S$, $\Phi^{-1} = (R_{z,\theta})^{-1} = R_{z,-\theta}$ $\therefore$ diffeomorphism

Ex: Sphere $S^2 = \{ (x, y, z) \mid x^2 + y^2 + z^2 = 1 \}$

Ellipsoid $E^2 = \{ (x, y, z) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \}$, $a, b, c \neq 0$

$\Phi : S^2 \to E^2$, $(x, y, z) \to (ax, by, cz)$, $\Phi^{-1}(x, y, z) = (\frac{x}{a}, \frac{y}{b}, \frac{z}{c})$. Φ is diffeomorphism

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Note. Let S_1, S_2, S_3 be three regular surfaces. If $\phi: S_1 \to S_2$ and $\psi: S_2 \to S_3$ are diff. map and $p \in S_1$,

then $d(\phi \circ \psi)_p = d\psi_{\phi(p)} \circ d\phi_p$

<pf> $p \in S$, $d\phi_p : T_p(S_1) \to T_{\phi(p)}(S_2)$, $d\psi_{\phi(p)} : T_{\phi(p)}(S_2) \to T_{\psi\circ\phi(p)}(S_3)$

Take parametrizations for surfaces, (u,v) for S_1, (x,y) for S_2, (s,t) for S_1

write $\phi(u,v) = (\phi_1(u,v), \phi_2(u,v)) = (x,y)$, $d\phi_p = \frac{\partial\phi}{\partial u} \times \frac{\partial \phi}{\partial v} = \begin{pmatrix} \frac{\partial\phi_1}{\partial u} & \frac{\partial\phi_1}{\partial v} \\ \frac{\partial\phi_2}{\partial u} & \frac{\partial\phi_2}{\partial v} \end{pmatrix}$ ... A

$\psi_{\phi(p)}(x,y) = (\psi_1(x,y), \psi_2(x,y)) = (s,t)$, $d\psi_{\phi(p)} = \frac{\partial\psi}{\partial x} \times \frac{\partial \psi}{\partial y} = \begin{pmatrix} \frac{\partial\psi_1}{\partial x} & \frac{\partial\psi_1}{\partial y} \\ \frac{\partial\psi_2}{\partial x} & \frac{\partial\psi_2}{\partial y} \end{pmatrix}$ ... B

$$d\psi_{\phi(p)} \circ d\psi_p = AB = \begin{pmatrix} \frac{\partial\psi_1}{\partial x}\frac{\partial\phi_1}{\partial u} + \frac{\partial\psi_1}{\partial y}\frac{\partial\phi_2}{\partial u} & \frac{\partial\psi_1}{\partial x}\frac{\partial\phi_1}{\partial v} + \frac{\partial\psi_1}{\partial y}\frac{\partial\phi_2}{\partial v} \\ \frac{\partial\psi_2}{\partial x}\frac{\partial\phi_1}{\partial u} + \frac{\partial\psi_2}{\partial y}\frac{\partial\phi_2}{\partial u} & \frac{\partial\psi_2}{\partial x}\frac{\partial\phi_1}{\partial v} + \frac{\partial\psi_2}{\partial y}\frac{\partial\phi_2}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{\partial(\psi_1\circ\phi)}{\partial u} & \frac{\partial(\psi_1\circ\phi)}{\partial v} \\ \frac{\partial(\psi_2\circ\phi)}{\partial u} & \frac{\partial(\psi_2\circ\phi)}{\partial v} \end{pmatrix}$$

Def. Let S_1 and S_2 be two surfaces, S_1 and S_2 are called orthogonal at p, $p \in S_1 \cap S_2$

if their normal lines are perpendicular at p.

normal line : $\alpha_1(t) = \vec{x_1} + tN_1$, $\alpha_2(t) = \vec{x_2} + tN_2$, $\vec{x_i}$ parametrization of S_i, N_i normal vector of S_i

Ex : $S_1 : x^2+y^2+z^2 = 4y$, $S_2 : x^2+y^2+z^2 = 4z$, check S_1 and S_2 orthogonal ?

$f(x,y,z) = x^2+(y-2)^2+z^2$, $g(x,y,z) = x^2+y^2+(z-2)^2$ * $f^{-1}(2) = S_1$, $g^{-1}(2) = S_2$

$\nabla f = (2x, 2(y-2), 2z)$ is normal to S_1, $\nabla g = (2x, 2y, 2(z-2))$ is normal to S_2

$\langle \nabla f, \nabla g \rangle \overset{?}{=} 0$ $\quad 4x^2+4y^2-8y+4z^2-8z = 0$ $\quad \because S_1 = S_2 \iff y = z$

Remark. (1) A regular parametrized surface may intersect itself.

(2) Regular parametrized surfaces are of the useful to describle set Σ (surface)

which are regular surfaces expect for a finite number points and lines.

Prop. Let $\vec{x}: U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be a parametrized surface, let $q \in U$, then $\exists$ nbd V of q in $\mathbb{R}^2$ st.

$\vec{x}(V) \subseteq \mathbb{R}^3$ is a regular surface. (A parametrized surface is a local regular surface)

<pf> We need to check that $\vec{x}: V \to \vec{x}(V)$ is homeomorphism.

$\because \vec{x}$ is a regular parametrized surface $\Rightarrow d\vec{x}_q$ is non-singular, i.e. $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$, if $\vec{x}(u,v) = (x,y,z)$

consider the extension of $\vec{x}$ given by $\vec{x}: U \times \mathbb{R} \to \mathbb{R}^3$, $\vec{x}(u,v,t) = (x(u,v), y(u,v), z(u,v)+t)$

$\det(d\vec{x}_q) = \begin{pmatrix} x_u & x_v & x_t \\ y_u & y_v & y_t \\ z_u & z_v & z_t \end{pmatrix} \neq 0 \quad \because \frac{\partial(x,y)}{\partial(u,v)} \neq 0 \quad \therefore$ map is invertible.

By IFT, $\exists V_1$ nbd of q, W_1 nbd of $\vec{x}(q)$, $\vec{x}: V_1 \times \mathbb{R} \to W_1$ is diffeomorphism.

$\vec{x}$ is homeomorphism in a nbd of $q \times \{0\}$, $V = V_1 \cap U$, $\vec{x}|_{V \times \{0\}}$ is local homeomorphism.

Tangent Plane

Def. A tangent vector to a regular surface S at a point $p \in S$ is a tangent vector $\alpha'(0)$ of some

diff. parametrized curve.

If v is a tangent vector of S, $\exists \alpha: I \to \mathbb{R}^3$, $\alpha(0) = p$, $\alpha'(0) = v$. $T_p(S) = \{ v \in \mathbb{R}^3 \}$

Prop. Let $\vec{x}: U \subseteq \mathbb{R}^2 \to S$ be a parametrization of a regular surface S and let $q \in U$,

the vector space of dimension 2 given by $d\vec{x}_q(\mathbb{R}^2)$, $d\vec{x}_q(\mathbb{R}^2) = T_p(S)$, $\vec{x}(q) = p \in S$,

where $T_p(S) \equiv$ tangent plane to S at $\vec{x}(q) = p \in S$

<pf> ($\Rightarrow$) on U, we can take a straight line $\alpha(t) = (at + u_0, bt + v_0)$, where $(u_0, v_0) = \alpha(0) = q$

$\alpha'(t) = (a,b) \in \mathbb{R}^2$, $\alpha'(0) = (a,b)$, $d\vec{x}_q(\alpha'(0)) = \frac{d}{dt}(\vec{x} \circ \alpha)(t)\Big|_{t=0} \in T_p(S) \quad \therefore d\vec{x}_q(\mathbb{R}^2) \subseteq T_p(S)$

($\Leftarrow$) Any tangent vector can be obtained this way $T_p(S) \subseteq d\vec{x}_q(\mathbb{R}^2)$

$\because$ S is a regular surface $\therefore$ $\vec{x}$ is differentiable ($\beta: I \to S$, $\beta(0) = p$, $\beta'(0) = w$)

Let $w = \beta'(0) \in T_p(S)$ and $(\vec{x}^{-1} \circ \beta)(t)$ is differentiable in U, $\vec{x}(\vec{x}^{-1} \circ \beta) = \beta$

Take $\frac{d}{dt}(\vec{x} \circ (\vec{x}^{-1} \circ \beta)(t)) = \frac{d}{dt}\beta \Rightarrow d\vec{x}_q(\vec{x}^{-1} \circ \beta)' = \beta'(0) = w$, where $w \in d\vec{x}_q(\mathbb{R}^2)$

Note. (1) $T_p(S)$ is a vector space. (2) $T_p(S)$ is spanned by $\{\frac{\partial \vec{x}}{\partial u}, \frac{\partial \vec{x}}{\partial v}\} = \{\vec{x}_u, \vec{x}_v\}$

Prop. $f: U \subseteq \mathbb{R}^3 \to \mathbb{R}$ differentiable function, a is regular value of f.

$S \equiv f^{-1}(a)$ is a regular surface (By Prop 2). $T_p(S) \equiv \text{kernal}\{df_p: \mathbb{R}^3 \to \mathbb{R}\}$, $p \in U$

<pf> ($\Rightarrow$) $v \in T_p(S)$, $\exists\, \alpha: I \to S$, $\alpha(0) = p$, $\alpha'(0) = v$, $f(\alpha(t)) = a$, $\forall t$ ($\because S = f^{-1}(a)$)

$df_p(v) = \frac{d}{dt}(f \circ \alpha)(t)\big|_{t=0} = \frac{d}{dt}(f(\alpha(t)))\big|_{t=0} = 0$

($\Leftarrow$) By Dimensional Thm: $\dim(\mathbb{R}^3) = \dim\ker\{df_p\} + \dim(\mathbb{R}) \Rightarrow \dim\ker\{df_p\} = 2$

$T_p(S) \subseteq \text{kernal}\{df_p\}$, $\dim T_p(S) = 2 \Rightarrow T_p(S) = \text{kernal}\{df_p\}$

Ex: $S_c^2(r) = \{p \in \mathbb{R}^3 \mid |p-c|^2 = r^2\}$ sphere with center c, radius r

$f: \mathbb{R}^3 \to \mathbb{R}$, $f(p) = |p-c|^2$, $f^{-1}(r^2) = S_c^2(r)$ $\therefore T_p(S_c^2(r)) \equiv \text{kernal}\{df_p: \mathbb{R}^3 \to \mathbb{R}\}$

$df_p(v) = \frac{d}{dt}(f \circ \alpha)(t)\big|_{t=0} = \frac{d}{dt}(f(\alpha(t)))\big|_{t=0} = \frac{d}{dt}\langle \alpha(t) - c, \alpha(t) - c\rangle\big|_{t=0} = 2\langle v, p-c\rangle = 0$

$v \in T_p(S_c^2(r))$, v is perpendicular to $p - c$, $v \perp$ radius vector

Prop. The map $d\phi_p: T_p(S_1) \to T_{\phi(p)}(S_2)$ defined by $d\phi_p(w) = (\phi \circ \alpha)'(0)$ is a linear map.

and $(\phi \circ \alpha)'(0)$ is independent on the choice of α, where S_1, S_2 are regular surfaces

<pf> Let $\vec{x_1}(u,v)$ and $\vec{x_2}(u,v)$ be two parametrization of $p \in S_1$ and $\Phi(p) \in S_2$ respectively.

Suppose $\Phi(u,v) = (\phi_1(u(t),v(t)), \phi_2(u(t),v(t))) = (\bar{u},\bar{v})$

Take $\alpha(t) = (u(t),v(t))$, $\alpha: (-\varepsilon,\varepsilon) \to S_1$, then $\alpha'(t) = (u'(t), v'(t))$

$\beta(t) = (\Phi \circ \alpha)(t) = (\phi_1(u(t),v(t)), \phi_2(u(t),v(t)))$, $\beta'(t)|_{t=0} = (\Phi \circ \alpha)'(t)|_{t=0} = \left(\frac{\partial \phi_1}{\partial u}u' + \frac{\partial \phi_1}{\partial v}v', \frac{\partial \phi_2}{\partial u}u' + \frac{\partial \phi_2}{\partial v}v'\right)\Big|_{t=0}$

$\therefore \beta'(0)$ depends only on Φ and the coordinate $(u'(0), v'(0))$ of w in the basis $\{\vec{x_u}, \vec{x_v}\}$

$\therefore d\Phi_p$ is a linear map of $T_p(S_1)$ into $T_{\Phi(p)}(S_2)$ whose matrix in the basis $\{\vec{x_u}, \vec{x_v}\}$ of $T_p(S_1)$, $\{\vec{x_{\bar{u}}}, \vec{x_{\bar{v}}}\}$ of $T_{\Phi(p)}($

Ex: Let $S^2 \subseteq \mathbb{R}^3$, $S^2 = \{(x,y,z) \mid x^2+y^2+z^2=1\}$, let $R_{z,\theta}: \mathbb{R}^3 \to \mathbb{R}$ be rotation of angle θ about z-axis

$p \in S^2$, compute $(dR_{z,\theta})_p(w)$, for $w \in T_p(S^2)$, let α be a curve st. $\alpha(0) = p$, $\alpha'(0) = w$

$\because R_{z,\theta}$ is a linear map on $\mathbb{R}^3$, $\frac{d}{dt}(R_{z,\theta} \circ \alpha)|_t = (R_{z,\theta}(\alpha(0)))\alpha'(0) = R_{z,\theta}(w)$, $dR_{z,\theta p}: T_p(S^2) \to T_p(S^2)$

Def. A mapping $\Phi: U \subseteq S_1 \to S_2$ is local diffeomorphism at $p \in U$,

if $\exists$ nbd $V \subseteq U$ of p st. $\Phi|_V$ is diffeomorphism onto an open set $\Phi(V) \subseteq S_2$.

Prop. If S_1 and S_2 are regular surfaces $\Phi: U \subseteq S_1 \to S_2$ is a differentiable map of an open set $U \subseteq S_1$

st. the differentiable map $d\Phi$ of Φ at $p \in U$ is isomorphism, then Φ is a local differentiable at p.

Def. A critical point p of a diff. function $f: S \to \mathbb{R}$ defined on a regular surface S at $p \in S$ st. $df_p = 0$

Ex: S: regular surface, $f: S \to \mathbb{R}$ is given by $f(p) = |p - p_0|^2$

If $v \in T_p(S)$, $\exists$ a diff. parametrized curve $\alpha: (-\varepsilon,\varepsilon) \to S$ s.t. $\alpha(0) = p$, $\alpha'(0) = v$

$df_p(v) = \frac{d}{dt}(f \circ \alpha)(t)|_{t=0} = \frac{d}{dt}(f(\alpha(t)))|_{t=0} = \frac{d}{dt}\langle \alpha(t) - p_0, \alpha(t) - p_0 \rangle|_{t=0} = 2\langle v, p - p_0 \rangle \cdots *$

If p is a critical point of f, LHS of $*$, $df_p = 0$ $\therefore$ RHS of $*$, $\langle v, \underset{\substack{\uparrow \\ \text{normal line of } S}}{p - p_0} \rangle = 0$

No. 8

Date 106 . 10 . 26

Surface of revolution

Take a regular surface in x-z plane, $C: \mathbb{R} \to \mathbb{R}^3$, $C(v) = (f(v), 0, g(v))$, $x = f(v) > 0$, $z = g(v) > 0$, $a < v < b$

$\vec{x}: U \subseteq \mathbb{R}^2 \to S \subseteq \mathbb{R}^3$ surface obtained by rotating the curve C about the z-axis,

where $U = \{(u,v) \mid 0 < u < 2\pi, a < v < b\}$, $\vec{x}(u,v) = (f(v)\cos u, f(v)\sin u, g(v))$ check (1) ✓ (2) (3)

$\vec{x}$ is 1-1 : $\vec{x}(u,v) = \vec{x}(u',v') \overset{?}{\Rightarrow} u = u', v = v'$. $\underset{①}{f(v)\cos u = f(v')\cos u'}$, $\underset{②}{f(v)\sin u = f(v')\sin u'}$, $\underset{③}{g(v) = g(v')}$

$①^2 + ②^2 \Rightarrow f^2(v) = f^2(v')$ $\because f(v) > 0$ $\therefore f(v) = f(v') \Rightarrow u = u'$ $\because C$ is a regular curve $\therefore ③ \Rightarrow v = v'$

$\vec{x}$ is continuous, $\vec{x}^{-1}$ exists and continuous (Homework)

check $d\vec{x}$ is nonsingular, i.e. $\frac{\partial \vec{x}}{\partial u} \times \frac{\partial \vec{x}}{\partial v} \neq 0$. $\frac{\partial \vec{x}}{\partial u} = (-f(v)\sin u, f(v)\cos u, 0)$, $\frac{\partial \vec{x}}{\partial v} = (f'(v)\cos u, f'(v)\sin u, g'(v))$

$\frac{\partial(x,y)}{\partial(u,v)} = -f(v)f'(v) \cdots ①$, $\frac{\partial(y,z)}{\partial(u,v)} = g'(v)f(v)\cos u \cdots ②$, $\frac{\partial(x,z)}{\partial(u,v)} = -f(v)g'(v)\sin u \cdots ③$

$\frac{\partial \vec{x}}{\partial u} \times \frac{\partial \vec{x}}{\partial v} = 0 \Leftrightarrow ① = ② = ③ = 0$. By ①, $\because f(v) > 0$ $\therefore f'(v) = 0$. By ②, ③, $f^2(v)(g'(v)) = 0$ $\therefore g'(v) = 0$

$\because C$ is a regular curve $\therefore f'(v) \neq 0$, $g'(v) \neq 0 \Rightarrow d\vec{x}$ is non-singular

Def. A parametrized surface $\vec{x}: U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ is differentiable map. The set $\vec{x}(U) \subseteq \mathbb{R}^3$ is called the trace of $\vec{x}$

We say $\vec{x}$ is regular if the differential $d\vec{x}_q: \mathbb{R}^2 \to \mathbb{R}^3$ is 1-1 (non-singular), $\forall q \in U$

Ex: Let $\alpha: I \to \mathbb{R}^3$ to a regular curve with nonzero curvature.

Define $\vec{x}(t,v) = \alpha(t) + v\alpha'(t)$, $\forall (t,v) \in I \times \mathbb{R}$, where $\vec{x}: U \to \mathbb{R}^3$, $U = \{(t,v) \mid v \neq 0\}$ differential ✓

$\frac{\partial \vec{x}}{\partial t} = \alpha'(t) + v\alpha''(t)$, $\frac{\partial \vec{x}}{\partial v} = \alpha'(t) \Rightarrow \frac{\partial \vec{x}}{\partial t} \times \frac{\partial \vec{x}}{\partial v} = \alpha'(t) \times \alpha'(t) + v\alpha'(t) \times \alpha''(t) \neq 0$ $\because k \neq 0$ $\therefore d\vec{x}$ is non-singular

$\therefore \vec{x}(U)$ is a regular parametrized surface which consists of two connected pieces whose common boundry is $\alpha(t)$.

The maps given by $\vec{x}_1: U_1 \to \mathbb{R}^3$, $U_1 = \{(t,v) \mid v > 0\}$, $\vec{x}_1(t,v) = \alpha(t) + v\alpha'(t)$

$\vec{x}_2: U_2 \to \mathbb{R}^3$, $U_2 = \{(t,v) \mid v < 0\}$

$\therefore \vec{x}_1(U_1)$ and $\vec{x}_2(U_2)$ are regular parametrized surface.

The first fundamental form ; Area

Def. $I_p : T_p(S) \to \mathbb{R}$ is defined by $I_p(w) = \langle w, w \rangle_p = |w|^2$, S is regular surface

The quadratic form I_p on $T_p(S)$ is called the first fundamental form of $S \subseteq \mathbb{R}^3$ at $p \in S$.

Let $\alpha(t)$ be a curve given by $\alpha(t) = \vec{x}(u(t), v(t))$ on S , write $\alpha(0) = P = \vec{x}(u_0, v_0)$

$I_p(\alpha'(t)) = \langle \vec{x_u} u' + \vec{x_v} v', \vec{x_u} u' + \vec{x_v} v' \rangle = (u')^2 \langle \vec{x_u}, \vec{x_u} \rangle + 2u'v' \langle \vec{x_u}, \vec{x_v} \rangle + (v')^2 \langle \vec{x_v}, \vec{x_v} \rangle$

$T_p(S) = \text{span}\{\vec{x_u}, \vec{x_v}\}$, call $E = \langle \vec{x_u}, \vec{x_u} \rangle$, $F = \langle \vec{x_u}, \vec{x_v} \rangle$, $G = \langle \vec{x_v}, \vec{x_v} \rangle$

$* = (u')^2 E + 2u'v' F + (v')^2 G$, E, F, G : coeff. of 1st fundamental form

In general, $w = a\vec{x_u} + b\vec{x_v}$ is a tangent vector at p. $I_p(w) = a^2 E + 2ab F + b^2 G$

Ex : compute the first fundamental form of sphere

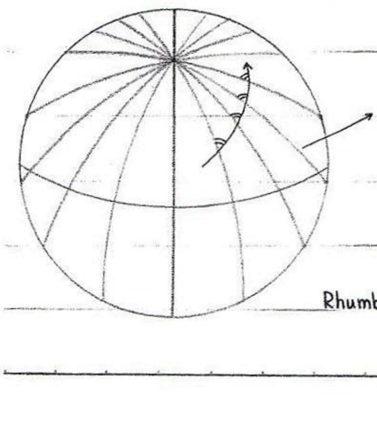
$\vec{X}(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. $T_p(S^2) = \text{span}\{\vec{x_\theta}, \vec{x_\phi}\}$

$\vec{x_\theta} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$, $\vec{x_\phi} = (-\sin\theta\sin\phi, \sin\theta\cos\phi, 0)$

$E = \langle \vec{x_\theta}, \vec{x_\theta} \rangle = 1$, $F = \langle \vec{x_\theta}, \vec{x_\phi} \rangle = 0$, $G = \langle \vec{x_\phi}, \vec{x_\phi} \rangle = \sin^2\theta$

$\therefore v = a\vec{x_\theta} + b\vec{x_\phi}$, $I_p(v) = a^2 + b^2\sin^2\theta$, $\forall\ v \in T_p(S)$

Hw : $x(u, v) = ((a + r\cos u)\cos v, (a + r\cos u)\sin v, r\sin u)$, $0 < u < 2\pi$, $0 < v < 2\pi$



meridiam 經線

Rhumb line (Loxodromic line) : A line on a sphere that cuts all meridiam at the same angle .

No. 6

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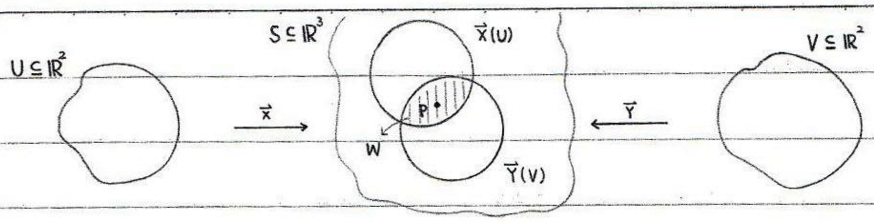


<pf> $h = \vec{x}^{-1} \circ \vec{Y}$ is homeomorphism $\therefore$ h is 1-1, onto, continuous and h^{-1} exists, continuous. $h^{-1} = \vec{Y}^{-1} \circ \vec{x}$

Let $r \in \vec{Y}^{-1}(W)$, $q \in \vec{x}^{-1}(W)$ st. $h(r) = q$. Since $\vec{x}$ is a parametrization of S. Assume that $\frac{\partial(x,y)}{\partial(u,v)}\Big|_q \neq 0$

Extend $\vec{x}$ to $F: U \times \mathbb{R} \to \mathbb{R}^3$ by $F(u,v,t) = (x(u,v), y(u,v), z(u,v)+t)$ $\therefore$ F is differentiable

$dF(q) = \begin{pmatrix} x_u & x_v & x_t \\ y_u & y_v & y_t \\ z_u & z_v & z_t \end{pmatrix}$, $\det(dF(q)) = \begin{vmatrix} x_u & x_v & 0 \\ y_u & y_v & 0 \\ z_u & z_v & 1 \end{vmatrix}_q = x_u y_v - x_v y_u|_q \neq 0$ $\therefore$ F is isomorphism.

By IFT, F^{-1} exists and diff, $\exists$ nbd U' of q, W' of $F(q)$ respectively $F^{-1}: W' \to U'$ differentiable

Clearly, $F^{-1}|_{W \cap S} = \vec{x}^{-1}$ ($\because F|_{U \times \{0\}} = \vec{x}$), hence $h = \vec{x}^{-1} \circ \vec{Y}$ is the same as $F^{-1} \circ \vec{Y}$ restrict on $\vec{Y}^{-1}(W' \cap S)$

$\therefore$ h is differentiable, the similar reason for h^{-1} is differentiable.

Def. $f: V \subseteq S \to \mathbb{R}$ is differentiable at $p \in S$, if for some parametrization of S,

$\vec{x}: U \to S \subseteq \mathbb{R}^3$ with $\vec{x}(U) \subseteq V$, $f \circ \vec{x}: U \to \mathbb{R}$ is differentiable at $\vec{x}^{-1}(p)$

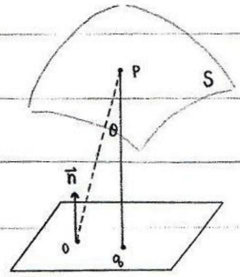
In particular, f is differentiable on V, if it is differentiable at every point of V.

Observe that if $f: W \subseteq \mathbb{R}^3 \to \mathbb{R}$ is differentiable, then $f|_S$ is differentiable. ($\exists\, \vec{x}$ of S st. $f \circ \vec{x}$ is diff.)

Ex: The height function given by a normal direction $\vec{n}$, with $|\vec{n}| = 1$

$f: \mathbb{R}^3 \to \mathbb{R}$, $p \in \mathbb{R}^3$, $f(p) = p \cdot \vec{n}$ differentiable, $\overline{Pq} = \overline{OP} \cdot \cos\theta = \overline{OP} \cdot \vec{n}$

$f: S \subseteq \mathbb{R}^3 \to \mathbb{R}$, $p \in S$, $f(p) = p \cdot \vec{n}$ differentiable



Ex: Distance square function $f: \mathbb{R}^3 \to \mathbb{R}$, $d^2 = f(p) = |p-q|^2 = \langle p-q, p-q \rangle$ $\therefore f: S \to \mathbb{R}$ differentiable

$d = g(p) = |p-q| = \langle p-q, p-q \rangle^{\frac{1}{2}}$, $g: \mathbb{R}^3 \setminus \{q\} \to \mathbb{R}$ differentiable $\therefore g: S \setminus \{q\} \to \mathbb{R}$ differentiable

Ex : Evalue the area of the sphere S^2, $\vec{x}(\theta,\phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, $Q = \{(\theta,\phi) \mid 0 < \theta, \phi < \frac{\pi}{2}\}$

$A(\vec{x}(Q)) = \frac{1}{8} A(S^2) = \iint_Q \sqrt{EG - F^2}\, du dv = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin\theta \, d\theta \, d\phi = \frac{\pi}{2}$ $\therefore A(S^2) = 4\pi$

Suppose that S_1 and S_2 are surfaces. We say that they are locally isometric,

if for each $p \in S_1$, $\exists \vec{x_1} : U \to S_1$ with $\vec{x_1}(u_0, v_0) = P$ and $\vec{x_2} : U \to S_2$ st.

$I_p(w_1) = I_p(w_2)$, whenever $P_1 = \vec{x_1}(u,v)$, $P_2 = \vec{x_2}(u,v)$, for $(u,v) \in U$

i.e. if $f = \vec{x_2} \circ \vec{x_1}^{-1} : \vec{x_1}(U) \to \vec{x_2}(U)$ is 1-1 correspond that preserve the 1 st. f.f. and distance preservary.

Ex : S_1 : plane, $X(u,v) = (u, v, 0)$ S_2 : cylinder, $\tilde{X}(u,v) = (\cos u, \sin u, v)$

$X_u = (1, 0, 0)$, $X_v = (0, 1, 0)$, $\tilde{X}_u = (-\sin u, \cos u, 0)$, $\tilde{X}_v = (0, 0, 1)$

$E = \langle X_u, X_u \rangle = 1 = \tilde{E} = \langle \tilde{X}_u, \tilde{X}_u \rangle$, $F = \langle X_u, X_v \rangle = 0 = \tilde{F}$, $G = \langle X_v, X_v \rangle = 1 = \tilde{G}$

$\therefore S_1$ is locally isometric to S_2, but S_1 and S_2 are not globally isometric

$\because$ The point far away in S_1 can be closed in S_2

Orientation of a surface

$\{e_1, e_2\}$ is a positive orientation basis, $\{e_1', e_2'\}$ is another positive orientation basis

$e_1' \times e_2' = \det(A)(e_1 \times e_2)$, $\{e_1, e_2\}$ and $\{e_1', e_2'\}$ have same orientation $\Leftrightarrow \det(A) > 0$

on $S \subseteq \mathbb{R}^3$, $T_p(S)$ has a basis $\{X_u, X_v\}$, so that $\{X_u, X_v\}$ gives an orientation on $\vec{X}(U)$

If $\bar{X} : \bar{U} \to S$ is another parametrization of S. If $X(U) \cap \bar{X}(\bar{U}) \neq \phi$, $\bar{X}_{\bar{u}} \times \bar{X}_{\bar{v}} = \frac{\partial(u,v)}{\partial(\bar{u},\bar{v})} X_u \times X_v$

Hence $\{\bar{X}_{\bar{u}}, \bar{X}_{\bar{v}}\}$ and $\{X_u, X_v\}$ give same orientation $\Leftrightarrow \frac{\partial(u,v)}{\partial(\bar{u},\bar{v})} > 0$

No. 4

Date 106 10 17

Inverse Function Theorem :

Let $F: U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable mapping and suppose that at $p \in U$ the differential map

$dF_p : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism, i.e. $\det(dF_p) \neq 0$

Then $\exists$ nbd V of p in U and a nbd W of $F(p)$ in $\mathbb{R}^n$, $F: V \to W$ has differential inverse $F^{-1}: W \to V$

Quadric Surface : $Ax^2 + By^2 + Cz^2 + Dx + Ey + Fz + Gxy + Hyz + Ixz + J = 0$

let $A \equiv$ a constant 4×4 symmetric matrix $(A^t = A)$

Quadric Surface S is given by $S = \{ u \in \mathbb{R}^3 \mid (1 \ u^t) A_{4\times4} \binom{1}{u} = 0 \}$, u is a vector, $u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ column matrix

let $f: \mathbb{R}^3 \to \mathbb{R}$, $f(u) = (1 \ u^t) A \binom{1}{u}$, $f^{-1}(0) = S$. we want to show that 0 is a regular value of f.

$df: \mathbb{R}^3 \to \mathbb{R}$, take $p \in f^{-1}(0)$, $df_p(w) = ?$, where w is a vector

let α be a regular curve, $\alpha(0) = p$, $\alpha'(0) = w$, $df_p(w) = \frac{d}{dt}(f \circ \alpha)|_{t=0}$ $\because (f\circ\alpha)(t) = (1 \ \alpha(t)^t) A \binom{1}{\alpha(t)}$

$df_p(w) = \frac{d}{dt}(f\circ\alpha)|_{t=0} = (0 \ w^t) A \binom{1}{p} + (1 \ p^t) A \binom{0}{w} = 2(1 \ p^t) A \binom{0}{w}$

$df_p(w) = 0, \forall w \iff (1 \ p^t) A = (k, 0)$ $\because p \in f^{-1}(0) \Rightarrow k = 0$

regular value no solutions

0 is a critical value $\iff (1 \ p^t) A = 0$ has solutions

Ex : Hyperboloid given by $S = \{ (x,y,z) \in \mathbb{R}^3 \mid -x^2 - y^2 + z^2 = 1 \}$

(1) check S is a regular surface : let $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = -x^2 - y^2 + z^2$, then $S = f^{-1}(1) = \{ (x,y,z) \mid f(x,y,z) = 1 \}$

$f_x = -2x$, $f_y = -2y$, $f_z = 2z$, they are not equal to zero simutanously unless $x=0$, $y=0$, $z=0$

However, $(0,0,0)$ is a critical point of f, 1 is a regular value of f, by Prop 2.

(2) find the parametrization of S : S can be represented as $z^2 = 1 + x^2 + y^2$, has two sheets : $\pm\sqrt{1+x^2+y^2}$

$\therefore$, $\vec{x_1}(x,y) = (x, y, \sqrt{1+x^2+y^2})$, $\vec{x_2}(x,y) = (x, y, -\sqrt{1+x^2+y^2})$ differentiable ✓ homeomorphism ✓

$d\vec{x_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \square & \triangle \end{pmatrix}$, $\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq 0$ $d\vec{x_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -\square & -\triangle \end{pmatrix}$, $\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq 0$

Mobius band : one of parametrization function $X(u,v) = \left((1-\frac{1}{2}v\sin(\frac{u}{2}))\sin u,\ (1-\frac{1}{2}v\sin(\frac{u}{2}))\cos u,\ \frac{1}{2}\cos(\frac{u}{2}) \right)$

$U = \{ (u,v) \mid 0 < u < 2\pi,\ -\pi < v < \pi \}$

let $\bar{u} = u - \frac{\pi}{2}$, $\bar{X}(\bar{u},\bar{v}) = \left((1-\frac{1}{2}\bar{v}\sin(\frac{\bar{u}+\frac{\pi}{2}}{2})\cos\bar{u},\ -(1-\frac{1}{2}\bar{v}\sin(\frac{\bar{u}+\frac{\pi}{2}}{2})\sin\bar{u},\ \frac{1}{2}\bar{v}\cos(\frac{\bar{u}+\frac{\pi}{2}}{2}))\right)$

Prop. If a regular surface is given by $S \equiv \{ (x,y,z) \mid f(x,y,z) = a \}$, where $f: U \subseteq \mathbb{R}^3 \to \mathbb{R}$ is diff.

and a is a regular value of f, then S is orientiable.

<pf> We need to find a global define unit normal vector field.

Note that $\frac{\nabla f}{|\nabla f|}$ is a unit normal vector field.

Assume that $\nabla f \neq 0$ which is the case, because a is a regular value.

Let $\alpha(t)$ be a regular curve on S, $\alpha(t) = (x(t), y(t), z(t))$, $\alpha'(t) = (\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt})$

$\alpha(0) = (x(0), y(0), z(0)) = p$, $f(\alpha(t)) = a$, take derivative w.r.t t, $t \to 0$

$f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt} = 0 \Big|_{t=0} \Rightarrow \nabla f \perp \alpha'$ at p

$N = \frac{\nabla f}{|\nabla f|}$ $\because f$ is C^∞ $\therefore N \in C^\infty$, $|N| = 1$ $\Rightarrow$ by prop. S done !!

No. 2

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Ex : $U = \{(\theta,\phi) \mid 0<\theta<\pi,\ 0<\phi,\ 2\pi\}$, $\vec{x}: U \to \mathbb{R}^3$, $\vec{x}(\theta,\phi) = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$, $\vec{x}(U) \subseteq S^2$

(1) $\vec{x}$ is differentiable ✓ (2) $\vec{x}$ is homeomorphism ✓ (3) $d\vec{x}_q$ is non-singular

(3) $\frac{\partial \vec{x}}{\partial\theta} = (\cos\theta\cos\phi,\ \cos\theta\sin\phi,\ -\sin\theta)$, $\frac{\partial \vec{x}}{\partial\phi} = (-\sin\theta\sin\phi,\ \sin\theta\cos\phi,\ 0)$

$\frac{\partial(x,y)}{\partial(\theta,\phi)} = \cos\theta\sin\theta$, $\frac{\partial(y,z)}{\partial(\theta,\phi)} = \sin^2\theta\cos\phi$, $\frac{\partial(x,z)}{\partial(\theta,\phi)} = -\sin^2\theta\sin\phi$ $\therefore d\vec{x}_q$ is non-singular

$\vec{x}_2 = \vec{x}(\theta,\phi)$, $U' = \{(\theta,\phi) \mid 0<\theta<\pi,\ -\pi<\phi<\pi\} \Rightarrow \vec{x}(U) \cup \vec{x}_2(U') = S^2 - \{(0,0,1),(0,0,-1)\}$

$\therefore S^2 - \{(0,0,1),(0,0,-1)\}$ is a regular surface

Ex : $S^2 = \{(x,y,z) \mid x^2+y^2+z^2=1\}$, $\vec{x}: U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$, $\vec{x}_1(u,v) = (u,v,\sqrt{1-u^2-v^2})$, $U = \{(u,v) \mid u^2+v^2<1\}$

(1) $\vec{x}_1$ is differentiable : $\frac{\partial x}{\partial u} = \frac{\partial y}{\partial v} = 1$, $\frac{\partial x}{\partial v} = \frac{\partial y}{\partial u} = 0$, $\frac{\partial z}{\partial u} = -\frac{u}{\sqrt{1-u^2-v^2}}$, $\frac{\partial z}{\partial v} = -\frac{v}{\sqrt{1-u^2-v^2}}$

(2) $\vec{x}_1$ is homeomorphism : 1-1, onto, continuous. $\vec{x}_1(u,v) = \vec{x}_1(u',v') \Rightarrow u=u',\ v=v'$

$\vec{x}_1^{-1}$ exists, continuous ($\because \vec{x}^{-1}$ is the restriction of projection $(x,y,z)\to(x,y)$ to $\vec{x}_1(U)$)

(3) $\frac{\partial\vec{x}_1}{\partial u} \times \frac{\partial\vec{x}_1}{\partial v} \neq 0$? $\because \frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix}1&0\\0&1\end{pmatrix} \neq 0$ $\therefore d\vec{x}_1$ is non-singular

$\vec{x}_1(U)$ is upper hormisphere ; $\vec{x}_2: U \to \mathbb{R}^3$, $\vec{x}_2(u,v) = (u,v,-\sqrt{1-u^2-v^2})$ lower hormisphere

$\vec{x}_1(U) \cup \vec{x}_2(U) = S^2 \setminus E$, $E = \{(x,y,z) \mid x^2+y^2=1,\ z=0\}$

$\vec{x}_3(u,v) = (u,\sqrt{1-u^2-v^2},v)$, $\vec{x}_4(u,v) = (u,-\sqrt{\ },v)$, $\vec{x}_5(u,v) = (\sqrt{\ },u,v)$, $\vec{x}_6(u,v) = (-\sqrt{\ },u,v)$

$\bigcup_{i=1}^{6} \vec{x}_i(U) = S^2$ $\therefore S^2$ is a regular surface

Prop 1. If $f: U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ is a differentiable function on an open set $U \subseteq \mathbb{R}^2$

then the graph of f is a regular surface, i.e. $\{(x,y,f(x,y)) \mid x\in U,\ y\in U\}$

<pf> Let $\vec{x}: U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$, define $\vec{x}(u,v) = (u,v,f(u,v))$

(1) $\vec{x}$ is differentiable ✓

(2) $\vec{x}$ is 1-1, onto, $\vec{x}^{-1}$ exists, continuous, $\vec{x}^{-1}$ is the restriction to the graph of projection $(x,y,z)\to(x,y)$

(3) check $d\vec{x}_p$ is non-singular : $\frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix}1&0\\0&1\end{pmatrix} \neq 0$